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A Direct Integration Approach for the Estimation of Time-Dependent Boundary Heat Flux

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In a one-dimensional heat conduction domain with heated and insulated walls, an integral approach is proposed to estimate time-dependent boundary heat flux without internal measurements. It is assumed that the expression of the heat flux is not known *a priori*. Hence, the present inverse heat conduction problem is classified as a function estimation problem. The spatial temperature distribution is approximated as a third-order polynomial of position, whose four coefficients are determined from the heat fluxes and the temperatures at both ends at each measurement. After integrating the heat conduction equation over spatial and time domain, respectively, a simple and non-iterative recursive equation to estimate the time-dependent boundary heat flux is derived. Several examples are introduced to show the effectiveness of the present approach.

Key Words: Inverse Heat Conduction Problem, Integration Approach, Boundary Heat Flux

Nomenclature

C : Heat capacity per unit volume [J/m^3-K]
k : Thermal conductivity [W/m-K]
l : Length [m]
Q : Non-dimensionalized heat flux, q/q_0[-]
q : Boundary heat flux [W/m^2]
q_0 : Reference boundary heat flux [W/m^2]
R : Random number ranging $-0.5 \le R \le 0.5$ [-]
T : Non-dimensionalized temperature, $(T^* - T_i)/(q_0 l/k)$ [-]
T^* : Temperature [K]
T_i : Initial temperature [K]
T_L : Non-dimensionalized temperature at the left end [-]
T_R : Non-dimensionalized temperature at the right end [-]
t : Non-dimensionalized time, $\alpha t^*/l^2$ [-]
t^* : Time [s]
$u(t)$: Unit step function, $u(t)=0$ for $t<0$, $u(t)=1$ for $t \ge 0$ [-]
x : Non-dimensionalized coordinate, x^*/l [-]
x^* : Coordinate [m]

Greeks

α : Thermal diffusivity [m^2/s]
ΔT : Error level [-]

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1. Introduction

The inverse heat conduction problem (IHCP) has been received much attention in recent years since it has been widely used in practical engineering problems involving the estimation of surface or initial conditions (Hills and Hensel, 1986; Huang and Wang, 1999; Lee et al., 2000; Shenefelt et al., 2002) as well as thermal properties of a body (Jung et al., 2001; Kim, 2001) from known information like temperatures measured at the prescribed positions. There have been many methods to solve the IHCP and the majority of researchers use the approaches where the unknowns are determined to minimize the sum of squares of the differences between the measured and the computed temperatures at the selected spatial and/or temporal points. In general, the approaches adopt the iterative scheme like the Newton-Raphson method (Jung et al., 2001) and the conjugate gradient method (Huang and Wang, 1999), and the regularizations are implemented to mitigate the ill-posedness of IHCP.

This paper concerns the estimation of time-varying boundary heat flux in a one-dimensional heat conduction domain with heated and insulated walls. Typical examples include predictions of heat flux from calorimeter-type instrumentation (Hills and Hensel, 1986). Many IHCP algorithms have been proposed to estimate the time-dependent boundary heat flux. Hills and Hensel (1986) developed a space-marching finite-difference algorithm to estimate the boundary heat flux non-iteratively and a digital filter to handle noisy data densely spaced in time. As an iterative method, the conjugate gradient method has been widely used to many kinds of IHCPs including the determination of boundary heat flux (Huang and Wang, 1999). However, the conjugate gradient method is computationally intensive, and may require large amounts of memory (Shenefelt et al., 2002). The Kalman filter (Lee et al., 2000; Daouas and Radhouani, 2000) is also often used for IHCPs.

The present work addresses an efficient method for the estimation of the temporal boundary heat flux of a one-dimensional homogeneous heat conduction medium using boundary data only. This work adopts the integral approach, which is not the first attempt and was already used to estimate the temperature-dependent thermal conductivity and heat capacity (e.g. Huang and Özişik, 1991). It should be recalled that their approach assumed the unknowns (thermal conductivity and heat capacity) to be a linear function whose unknown coefficients were determined in a least-square sense, while the proposed approach does not require any assumption on the functional form of the unknown (boundary heat flux). Hence, the present inverse problem can be classified into a function estimation problem.

In a one-dimensional heat conduction domain with heated and insulated ends, to a good approximation, the spatial temperature distribution is modeled as a third-order function of position. Four time-dependent coefficients of the function of temperature distribution are determined from the heat fluxes imposed and the temperatures measured at both ends at each temporal measurement. As a consequence of the integration of the heat conduction equation over the spatial and temporal domain, a very simple recursive expression of the boundary heat flux is obtained. Various heating and/or cooling problems are introduced to verify the proposed approach. The effect of measurement error on the performance of the approach is also examined. In view of the fact that the present approach is non-iterative and does not require any regularization to some extent to alleviate the ill-posedness, which is pronounced with the contaminated measurement data, it has a peculiar difference from the previous studies.

2. Mathematical Model

Let us consider a one-dimensional homogeneous heat conduction medium as shown in Fig. 1. It is assumed that the left end is heated or cooled by a time-varying heat flux, $q(t)$, and the right is insulated. For the case of constant thermal conductivity k and heat capacity per unit volume C, the heat conduction equation reads

$$\frac{\partial T^*}{\partial t^*}=\alpha\frac{\partial^2 T^*}{\partial x^{*2}} \qquad x^*\in[0,\ l],\ t^*\in(0,\ \infty) \quad (1a)$$

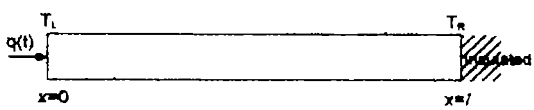


Fig. 1 Description of problem domain

which is subject to the initial and boundary conditions

$$T^*(t^*=0, x^*)=T_i \tag{1b}$$

$$-k\frac{\partial T^*}{\partial x^*}\Big|_{x=0}=q(t) \text{ and } \frac{\partial T^*}{\partial x^*}\Big|_{x^*=l}=0 \tag{1c}$$

With the definition of the non-dimensionalized variables

$$x=\frac{x^*}{l},\ t=\frac{at^*}{l^2},\quad T=\frac{k(T^*-T_i)}{q_0 l},\ \text{and } Q=\frac{q}{q_0} \tag{2}$$

the heat conduction equation will be

$$\frac{\partial T}{\partial t}=\frac{\partial^2 T}{\partial x^2} \qquad x\in[0, 1],\ t\in(0, \infty) \tag{3a}$$

which is subject to the initial and boundary conditions

$$T(t=0, x)=0 \tag{3b}$$

$$-\frac{\partial T}{\partial x}\Big|_{x=0}=Q(t) \text{ and } \frac{\partial T}{\partial x}\Big|_{x=1}=0 \tag{3c}$$

In this, q_0 denotes the reference boundary heat flux. Also, the temperatures at the ends, which can be known from the measurements, will be time-dependent.

$$T(t, x=0)=T_L(t) \text{ and } T(t, x=1)=T_R(t) \tag{4}$$

In order to apply the direct integration approach to the present inverse analysis, the heat conduction equation, Eq. (3), is integrated with respect to spatial and time coordinate :

$$\int_0^t\int_0^1\frac{\partial T}{\partial \tau}d\tau dx=\int_0^t\int_0^1\frac{\partial^2 T}{\partial x^2}dx d\tau \quad \text{or} \int_0^1 T dx=\int_0^t Q(\tau)d\tau \tag{5}$$

If the temperature distribution is known within the domain of interest one can construct a relation that heat flux Q should satisfy approximately. However, the temperature distribution is not known *a priori*. In this work, hence, we approximate the temperature distribution as a third-order polynomial with four time-dependent coefficients. If the temperatures at both ends and the heat flux at the left end are available, one can have

$$\begin{aligned} T(t, x)&=a_0(t)+a_1(t)x+a_2(t)x^2+a_3(t)x^3\\ &=(1-x^2)[(T_L(t)-T_R(t))(2x+1)\\ &\quad -Q(t)x]+T_R \end{aligned} \tag{6}$$

Inserting Eq. (6) into Eq. (5) gives

$$Q(t)+12\int_0^t Q(\tau)d\tau=6[T_L(t)-T_R(t)]+12T_R(t) \tag{7}$$

Considering the difference between the time step $t=t_i$ and $t=t_{i-1}$

$$Q_i-Q_{i-1}+12\int_{t_{i-1}}^{t_i}Q(\tau)dt=6(\Delta T_i-\Delta T_{i-1})+12(T_{R,i}-T_{R,i-1}) \tag{8}$$

will be obtained and finally a recursive expression of the boundary heat flux becomes

$$(1+6\Delta t_i)Q_i=(1-6\Delta t_i)Q_{i-1}+6(\Delta T_i-\Delta T_{i-1})+12[T_R(t_i)-T_R(t_{i-1})] \tag{9}$$

where $\Delta t_i=t_i-t_{i-1}$ and $\Delta T_i=T_L(t_i)-T_R(t_i)$.

3. Examples

In order to evaluate the proposed algorithm, seven examples are introduced. In Examples 1~5, the heat flux is assumed to be continuously varying. For considering abruptly changing situations, Examples 6 and 7 are introduced.

Example 1 : $Q(t)=0.1t$

Example 2 : $Q(t)=0.04t(10-t)$

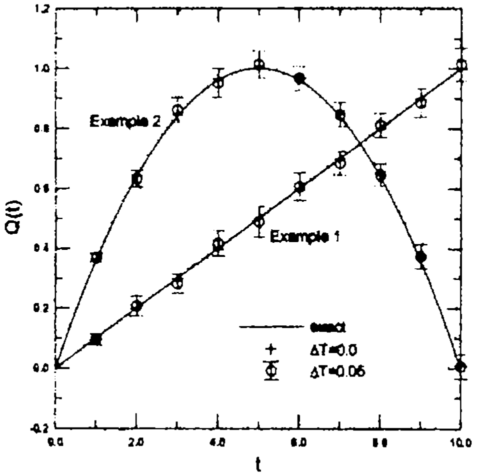
Example 3 : $Q(t)=24t^3-35t^2+13t$

Example 4 : $Q(t)=t\sin(10\pi t)$

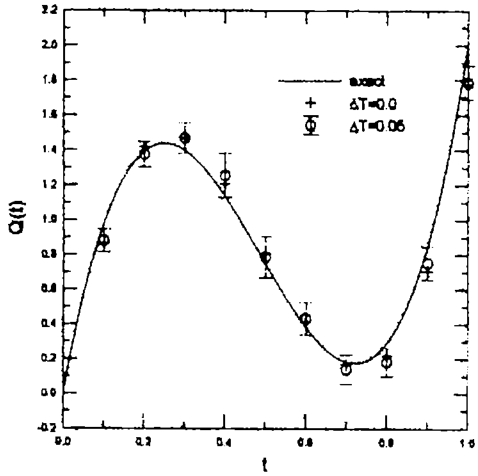
Example 5 : $Q(t)=2t[u(t)-u(t-0.5)]+2(1-t)[u(t)-u(t-0.5)]+2(t-2)[u(t)-u(t-0.5)]+2(3-t)[u(t)-u(t-0.5)]$

Example 6 : $Q(t)=u(t-0.05)-u(t-0.2)+u(t-0.35)-u(t-0.5)+u(t-0.65)-u(t-0.8)$

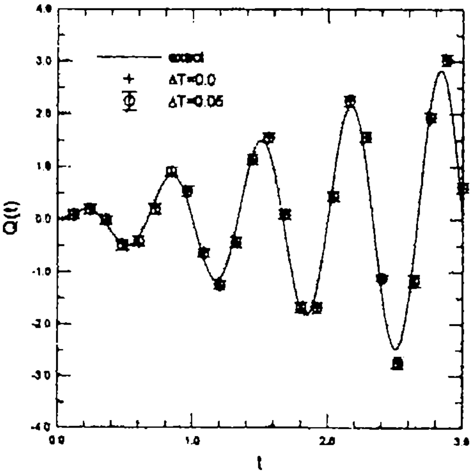
Example 7 : $Q(t)=4(t-0.3)[u(t-0.3)-u(t-0.5)]+4(0.7-t)[u(t-0.5)-u(t-0.6)]+0.4[u(t-0.6)-u(t-0.8)]+[(t-0.8)-u(t-1)]+0.2[u(t-1)-u(t-1.13)]+[0.2+0.8\sin(20(1.13-t))][u(t-1.13)-u(t-1.6)]$



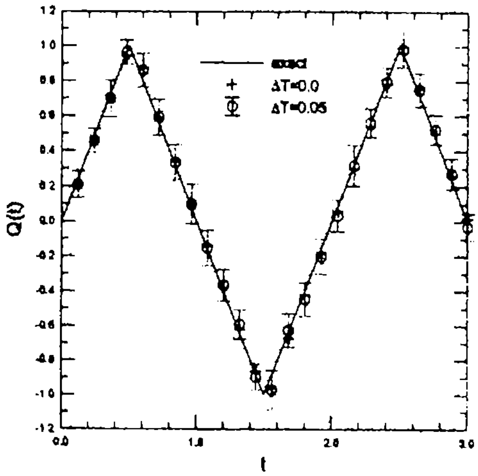
(a) Examples 1 and 2



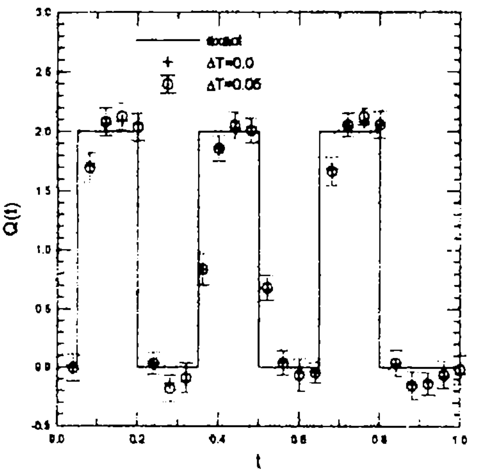
(b) Example 3



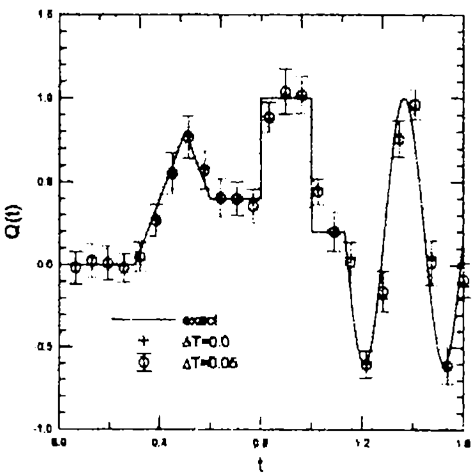
(c) Example 4



(d) Example 5



(e) Example 6



(f) Example 7

Fig. 2 Estimated boundary heat fluxes

In this, $u(t)$ is a unit step function defined as $u(t)=0$ for $t<0$ and $u(t)=1$ for $t\geq 0$.

Now, we will conduct numerical experiments to reconstruct time-varying heat flux with the synthesized boundary temperatures. The boundary temperatures of each example can be calculated easily from the exact solution of Eq. (3) (Hills and Hensel, 1986):

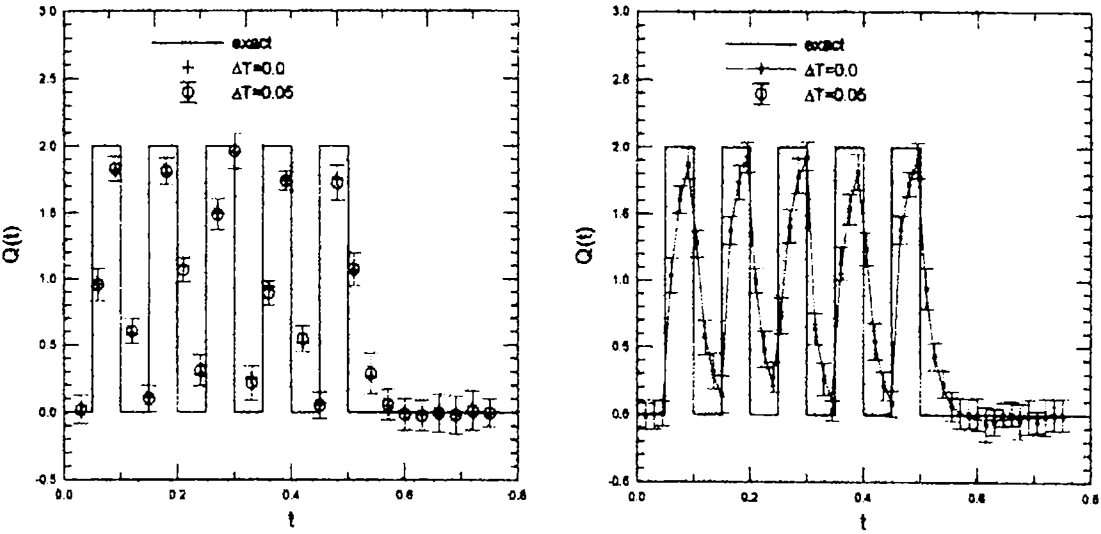
$$T(t,x)=\frac{3(1-x)^2-1}{6}Q(t)-2Q(t)\sum_{n=1}^{\infty}\frac{\cos(\lambda_n x)}{\lambda_n^2}+\int_0^t Q(\tau)d\tau+2\sum_{n=1}^{\infty}\cos(\lambda_n x)\int_0^t Q(\tau)\exp[-\lambda_n^2(t-\tau)]d\tau \tag{10}$$

where $\lambda_n=n\pi$. With exact values of $T_L(t)=T(t,0)$ and $T_R(t)=T(t,1)$, and from Eq. (9), one could readily estimate the boundary fluxes of Examples 1～7 as shown in Fig. 2. For Examples 1～3, 10 measurements in the temporal coordinate are used. As indicated in the figures, the proposed approach reproduces the variation of the boundary heat flux quite well even with small number of measurements. For Examples 4～7 where the heat flux is more fluctuating, 10 measurements give somewhat poor results. Thus, we increase the number of measurements to 25, which predicts the heat flux distribution in the temporal domain very satisfactorily. In order to investigate the effect of the number of measurements on the estimation, we also employ 50 measurements and it results in a little improvement. It should be noted that for the heat flux varying abruptly in time (see Examples 6 and 7) the proposed algorithm generates some delay in the estimates. The results show that the time lag is roughly estimated to be 0.1 in the dimensionless time, which is very comparable to that observed by other investigators (Hills and Hensel, 1986; Daouas and Radhouani, 2000). It would be noteworthy that the time lag confined within the short time interval and does not affect the successive estimation.

From the estimated results with error-free data, we can evaluate favorable features of the present approach. However, unless the approach is not tested with noisy data that are likely to be in reality, we could not assert the usefulness of the proposed model since methods that work well with perfect data can prove to be useless when real data are used. Now, we will consider the measurement noise by adding random error to the exact temperatures:

$$T_{measured}(t,x)=T_{exact}(t,x)+R\Delta T \tag{11}$$

where ΔT is the error level and R is a random number ranging $-0.5\leq R\leq 0.5$. In order to assess the performance of the present model when measured temperatures are contaminated by the error, we conduct the inverse estimation of the boundary heat flux with imposing the error level of $\Delta T=0.05$ on the exact boundary temperatures for all examples. The error level of $\Delta T=0.05$ corresponds to $0.05q_0 l/K$ in the unit of degree Kelvin and, for example, with $q_0=10$ kW/m², $k=100$ W/m-K (metal), and $l=0.1$ m the error level will be 0.5 K, which is sufficiently large in the practical temperature measurement. In order to clarify the statistical error in the inverse solution due to the error in the measurement of boundary temperatures, 20 sets of measurements are assumed. The statistical averages and the error bounds engaged in each inverse solution are illustrated in Fig. 2. The error bound represents the sample standard deviation. The estimated heat flux variations with the error-free data and with the contaminated data are presented in the same figure for the comparison. The results show that even with the measurement error of $\Delta T=0.05$ the proposed algorithm appears to track the oscillatory behavior of the true surface heat flux fairly well. Similar to the case without measurement errors, in the estimation of abruptly



(a) 25 measurements (b) 50 measurements

Fig. 3 Rectangular pulse-typed heat flux

varying heat fluxes, the reconstructed heat fluxes with the contaminated data also exhibit the same time lag. That is, the measurement error does not magnify the time lag.

In order to appreciate the performance of the proposed algorithm for an abrupt heat flux variation with a short period, we consider a periodic rectangular pulse with a period of 0.1. In this case, as shown in Fig. 3, 25 measurements in the temporal coordinate (i.e. $\Delta t=0.03$) seem to be insufficient to reconstruct the heat flux transients and the number of measurements is increased to 50 (i.e. $\Delta t=0.015$). With 50 measurements, the results show the time lag obviously even though the algorithm can estimate the magnitude and the period of the heat flux variation. From the aforementioned examples, we could draw some conclusions:

(1) The proposed algorithm gives good estimates for continuously varying heat fluxes.

(2) For abruptly varying heat fluxes, the algorithm shows some time lag, which can be roughly estimated to be 0.1.

4. Conclusions

A direct integration approach is developed to estimate the surface heat flux without measuring internal temperatures. In one-dimensional time-dependent heat conduction medium bounded by a heated and an insulated ends, the internal temperature distribution is approximated as a third-order polynomial, whose coefficients are expressed by two heat fluxes imposed and two temperatures measured at the ends. Integrating the heat conduction equation over the space and time domain, we obtain a simple algebraic recursive expression to estimate the temporal heat flux. Hence, the proposed method does not require any steps to solve the heat conduction equation nor iterations. In spite of the simple feature of the present model, the tested examples with and without measurement error show that the model gives excellent agreements with the true heat fluxes varying continuously in time. For abruptly varying heat fluxes, however, this model gives a time lag of about 0.1 in the dimensionless time. The proposed algorithm could also be used to generate the initial guess for more sophisticated schemes usually employing iterative procedures.

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